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A Van der Pol–Mathieu equation for the dynamics of dust grain charge in dusty plasmas

M Momeni[1], **I Kourakis**[2], **M Moslehi-Fard**[1] **and P K Shukla**[2]

[1] Faculty of Physics, Tabriz University, Tabriz 51664, Iran
[2] Institut für theoretische Physik IV, Fakultät für Physik und Astronomie, Ruhr-Universität Bochum, 44780 Bochum, Germany

E-mail: mehdy_momeny@yahoo.com and ioannis@tp4.rub.de

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Abstract
The chaotic profile of dust grain dynamics associated with dust-acoustic oscillations in a dusty plasma is considered. The collective behaviour of the dust plasma component is described via a multi-fluid model, comprising Boltzmann distributed electrons and ions, as well as an equation of continuity possessing a source term for the dust grains, the dust momentum and Poisson's equations. A Van der Pol–Mathieu-type nonlinear ordinary differential equation for the dust grain density dynamics is derived. The dynamical system is cast into an autonomous form by employing an averaging method. Critical stability boundaries for a particular trivial solution of the governing equation with varying parameters are specified. The equation is analysed to determine the resonance region, and finally numerically solved by using a fourth-order Runge–Kutta method. The presence of chaotic limit cycles is pointed out.

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1. Introduction

The properties of dusty plasmas have recently been attracting growing interest. A dusty (or complex) plasma, compared to an electron–ion plasma, i.e. a large ensemble of electrons and positive ions, is characterized by an additional charged component of micron or submicron-sized dust particulates. The dust grain charge, q_d, in contrast to the electron and ion charge, is not constant. When dust grains are immersed in a gaseous plasma, the charge residing on dust particles varies as a result of the flow of plasma particles onto their surface [1, 2]. Dust grains are considerably massive (typically a billion times heavier than protons), and their size ranges from nanometres to millimetres. The presence of massive charged dust particles in a plasma can drastically affect its dispersive and nonlinear properties [3, 4]. Dust grains are an important component of astrophysical systems where self-gravitational effects are dominant

[5, 6]. When the dust component is uncharged, the collective dynamics is governed entirely by gravitational interaction. However, dust grains surrounded by ionized media and radiative environments can become highly charged. Since both interactions are of long range, and the gravitational interaction is much weaker than the electromagnetic interaction, the latter can significantly influence the collective dynamics of a spatially extended self-gravitating system.

The collective behaviour of charged dust in a plasma is effectively described via a (multi-)fluid model. The presence of charged dust grains introduces new features in the nonlinear dynamical profile of dusty plasmas, some of which are absent in ordinary electron–ion plasmas. Various linear and nonlinear mechanisms involved in dusty plasmas give rise to different instabilities, whose elucidation is of great importance in understanding the origin of dust fluctuations. A recent study was devoted to a dust charging instability modelled via the chaotic behaviour of charged dust in the plasma [7]. A Van der Pol–Mathieu equation was introduced therein in order to model the dynamical behaviour of the dust grain charge. Generic nonlinear oscillator model differential equations of this kind can be studied by existing analytical methods [8, 9]. The Van der Pol (VdP) equation has been studied by many researchers [10–12]. Siewe [13] has investigated a system consisting of an extended Duffing–Van der Pol oscillator in which resonance and off-resonance oscillations are analysed using the multiple time scale method, while Maccari [14] introduced a new asymptotic perturbation method in search of an exact solution.

The present work aims at investigating the dynamical behaviour of charged dust grains near parametric resonance. Relying on a fluid model, which takes into account a source term for the dust grains, in addition to electrostatic, a Van der Pol–Mathieu nonlinear equation is shown to govern the dust grain dynamics. The equation is analysed in the vicinity of resonance, and finally numerically solved by using a fourth-order Runge–Kutta method.

2. Derivation of a Van der Pol–Mathieu equation for dust density

We consider an unmagnetized collisionless dusty plasma consisting of electrons (mass m_e, charge $-e$), ions (mass m_i, charge $+Z_i e$) and dust grains. We consider the mass of each dust grain m_d to be constant, while the dust grain charge is a time-dependent variable $q_d(t) = -Z_d(t)e$.

The cold inertial dust fluid density n_d and velocity v_d are governed by the density evolution equation

$$\frac{\partial n_d}{\partial t} + n_{d0}\frac{\partial u_d}{\partial z} = \alpha n_d - \frac{1}{3}\beta n_d^3, \tag{1}$$

where t and z are independent time and (one-dimensional) space variables. Here, the dust particle number density at equilibrium $n_{d,0}$ was employed to simplify the convective term, since the dust is assumed to be homogeneously distributed (namely $\partial n_d/\partial z \approx 0$). The coefficients α and β entering the source term on the right-hand side of equation (1) correspond to a rate of charged dust grain production (by electron absorption) and loss (due to three-body recombination, namely $X^+ + e^- + Z \to X^* + Z$), respectively. The dust momentum equation reads

$$\frac{\partial u_d}{\partial t} = -\frac{q_d}{m_d}\frac{\partial \phi}{\partial z}. \tag{2}$$

The electric potential ϕ is determined by the Poisson equation

$$\frac{\partial^2 \phi}{\partial z^2} = -4\pi e(Z_i n_i - n_e - Z_d n_d), \tag{3}$$

where n_i and n_e denote the ion and electron number density, respectively. At equilibrium, we have $Z_i n_{i0} - n_{e0} - Z_d n_{d0} = 0$, where n_{s0}, for $s = i(e)$, denotes the ion (electron) particle number density at equilibrium, respectively. We shall assume a harmonic potential variation in space, characterized by a wave length $\lambda \equiv 2\pi/k$ (and a wave number k), i.e. $\phi(z,t) = \hat{\phi}(t)\exp(ikz)$, so that $\partial^2\phi/\partial z^2 = -k^2\phi$.

The electrons and ions are assumed to be in local thermodynamic equilibrium, so their number densities, n_e and n_i, obey a Boltzmann distribution, namely

$$n_e = n_{e0}\exp\left(\frac{e\phi}{k_B T_e}\right), \tag{4}$$

and

$$n_i = n_{i0}\exp\left(\frac{-Z_i e\phi}{k_B T_i}\right), \tag{5}$$

where k_B is the Boltzmann constant and T_e (T_i) denotes the electron (ion) temperature.

By differentiating equations (1) and (2) with respect to time and space, respectively, in order to eliminate the velocity u_d, we obtain the equation

$$\frac{\partial^2 n_d}{\partial t^2} - \left(\alpha - \beta n_d^2\right)\frac{\partial n_d}{\partial t} = \frac{n_{d0}q_d}{m_d}\frac{\partial^2\phi}{\partial z^2}. \tag{6}$$

For $e\phi/k_B T_e \ll 1$ and $Z_i e\phi/k_B T_i \ll 1$, we can approximate n_e and n_i as

$$\frac{n_e}{n_{e0}} \approx 1 + \frac{e\phi}{k_B T_e} + \frac{1}{2}\left(\frac{e\phi}{k_B T_e}\right)^2 + \cdots \tag{7}$$

and

$$\frac{n_i}{n_{i0}} \approx 1 - \frac{Z_i e\phi}{k_B T_i} + \frac{1}{2}\left(\frac{Z_i e\phi}{k_B T_i}\right)^2 + \cdots. \tag{8}$$

Combining (7) and (8) into equation (3), and taking into account terms up to first order, we obtain

$$\frac{\partial^2\phi}{\partial z^2} \approx -\frac{4\pi q_d k^2}{k^2 + k_D^2} n_d, \tag{9}$$

where we have defined the Debye wave number $k_D \equiv \lambda_{\mathrm{Deff}}^{-1} = \left(\lambda_{De}^{-2} + \lambda_{Di}^{-2}\right)^{1/2}$, where $\lambda_{De} = (k_B T_e/4\pi n_{e0}e^2)^{\frac{1}{2}}$ and $\lambda_{Di} = \left(k_B T_i/4\pi n_{i0} Z_i^2 e^2\right)^{\frac{1}{2}}$ are the electron and ion Debye radii, respectively. Note that only the first-order contributions in expansions (7) and (8) have been retained to derive equation (9). Also, the dust component was assumed to be small (namely $Z_d n_{d,0} \ll Z_i n_{i,0}, n_{e,0}$, so we have set $Z_i n_{i,0} - n_{e,0} \approx 0$); this is in fact a realistic assumption, in naturally occurring dusty plasmas. We see that the inertialess electrons and ions affect the propagation of dust acoustic waves [15] via a dynamical charge balance. In the long Debye-wavelength limit $\lambda \ll \lambda_{\mathrm{Deff}}$, namely $k \gg k_D$, one may neglect the electric potential variation effect in equation (9).

By substituting equation (9) into equation (6), we can eliminate the potential ϕ. We shall assume, for analytical tractability, that the fluctuating dust charge varies in time as $q_d = q_{d0}(1 + h\cos\gamma t)^{1/2}$, where the (small) parameter h and the frequency γ are real constants. We thus finally obtain a closed evolution equation for the dust density

$$\frac{\mathrm{d}^2 n_d}{\mathrm{d}t^2} - \left(\alpha - \beta n_d^2\right)\frac{\mathrm{d}n_d}{\mathrm{d}t} + \omega_0^2(1 + h\cos\gamma t)n_d = 0, \tag{10}$$

where we have defined the characteristic oscillation frequency $\omega_0 = \omega_{pd}k\big/\left(k^2 + k_D^2\right)^{1/2}$ ($\approx\omega_{pd}$ in the long-wavelength limit), and the dust plasma frequency $\omega_{pd} = \left(4\pi n_{d0}q_{d0}^2\big/m_d\right)^{1/2}$. The dust density was assumed to be uniform in space. For our purposes, it is appropriate to cast the latter equation in a reduced (dimensionless) form. Defining the dimensionless time and density variables $\tilde{t} = \omega_0 t$ and $x = n_d/n_{d0}$, as well as the parameters $\tilde{\alpha} = \alpha/\omega_0$, $\tilde{\beta} = \beta n_{d0}^2 / \omega_0$ and $\tilde{\gamma} = \gamma/\omega_0$, we obtain

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\tilde{t}^2} - (\tilde{\alpha} - \tilde{\beta} x^2)\frac{\mathrm{d}x}{\mathrm{d}\tilde{t}} + \tilde{\omega}_0^2 (1 + h\cos\tilde{\gamma}\tilde{t})x = 0. \tag{11}$$

The dimensionless parameter $\tilde{\omega}_0$, obviously equal to unity, is retained in the following for 'book-keeping' purposes (to pin point the role of the plasma frequency in the forthcoming analysis, where appropriate). The reduced evolution equation (11) will be the object of the study which follows, so we shall keep in mind that all quantities henceforth refer to the physical quantities defined and appearing in (11); nevertheless, the tilde will be dropped everywhere.

The second term on the left-hand side of equation (11) is the characteristic of the Van der Pol (VdP) nonlinear oscillator model equation; indeed, the VdP equation, which generically describes a self-sustained nonlinear oscillation, is exactly recovered for $h = 0$. On the other hand, for $\alpha = \beta = 0$, one recovers a Mathieu-type equation, which describes a parametric-type oscillation. The ordinary differential equation (ODE) (11) is a hybrid equation, combining the features of the Van der Pol and the Mathieu equations.

The Van der Pol equation may be viewed as the fundamental example of a nonlinear ordinary differential equation. It possesses a periodic oscillatory solution, which is a periodic attractor. Every nontrivial solution tends to this periodic solution, a property that no linear flow can present. On the other hand, in order for a periodic solution to be viable, some related stability property must be satisfied. In contrast to this physical picture, an oscillatory dynamical system may be subject to an external force which changes the oscillation period parametrically, as in the Mathieu equation. In parametric resonance, where the oscillation parameters depend on time, if the initial condition of the velocity and position is zero, the system may be stable, in contrast to the case in ordinary resonance where the oscillation amplitude increases with time, even with a zero initial condition. The Van der Pol–Mathieu equation (11) will be analysed via perturbation theory, and will then be investigated numerically.

3. Qualitative analysis: the averaging method

In this section, we shall investigate the dynamical behaviour of the nonlinear Van der Pol–Mathieu (VdPM) oscillator, with the effect of parametric resonance. The basic equation (11) can be expressed as

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} - (\alpha - \beta x^2)\frac{\mathrm{d}x}{\mathrm{d}t} + \omega^2(t)x = 0, \tag{12}$$

where $\omega^2(t) = \omega_0^2(1+h\cos\gamma t)$ is the (reduced) time-dependent (square) oscillation frequency function.

3.1. Reduction to a pair of amplitude evolution ODE's

Since a parametric resonance is stronger for a frequency $\omega(t)$ nearly twice the eigenfrequency ω_0 (see e.g. [8, section 27]), we shall consider the parametric excitation frequency to be $\gamma = 2\omega_0 + \epsilon$, where $\epsilon \ll 1$ is a (small) real parameter.

We assume a solution given by the ansatz

$$x = a(t)\cos\left(\omega_0 + \frac{\epsilon}{2}\right)t + b(t)\sin\left(\omega_0 + \frac{\epsilon}{2}\right)t, \tag{13}$$

where the (real) coefficients a and b vary *slowly* with time.

Substituting equation (13) into (11) and keeping only first-order terms ϵ and h, we obtain the system of equations

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{\alpha}{2}a - \frac{b}{2}\left(\epsilon + \frac{h\omega_0}{2}\right) - \frac{\beta}{8}(a^3 + ab^2) \equiv f(a,b), \tag{14}$$

and

$$\frac{\mathrm{d}b}{\mathrm{d}t} = \frac{\alpha}{2}b + \frac{a}{2}\left(\epsilon - \frac{h\omega_0}{2}\right) - \frac{\beta}{8}(b^3 + a^2 b) \equiv g(a,b). \tag{15}$$

Equations (14) and (15) represent a system of first order, autonomous, ordinary differential equations, governing the amplitudes of the approximate solution expressed in (13). We note that equations (14) and (15) are invariant under the transformation $(a, b) \to (-a, -b)$.

3.2. Stability analysis

Relying on equations (14) and (15), we may determine the steady-state solution (defining equilibrium) and, possibly, periodic solutions around it, in addition to their stability profile. First, we note that the range of values of the small parameter ϵ is restricted. Assuming that the coefficients a and b are small and vary as $a \sim \exp(st)$ and $b \sim \exp(st)$, one obtains (upon linearizing) the relation

$$\left(s - \frac{\alpha}{2}\right)^2 = \frac{1}{4}\left[\left(\frac{h\omega_0}{2}\right)^2 - \epsilon^2\right], \tag{16}$$

which leads to the reality condition

$$|\epsilon| < \frac{|h|\omega_0}{2}. \tag{17}$$

3.3. Initial equilibrium solution

It is easily deduced from equations (14) and (15) that $a = b = 0$ (namely $x = 0$) is an equilibrium solution, which determines a *fixed point*. The stability of the fixed point is determined by the eigenvalues of the Jacobian matrix of the vector fields in equations (14) and (15). The characteristic polynomial reads

$$p(\lambda) = \lambda^2 - \alpha\lambda + \frac{\alpha^2}{4} + \frac{1}{4}\left(\epsilon^2 - \frac{h^2\omega_0^2}{4}\right). \tag{18}$$

From equation (18), it is clear that the Jacobian matrix of the vector field at the initial equilibrium solution has two complex (conjugate, to one another) eigenvalues, λ_1 and λ_2, namely

$$\lambda_{1,2} = A \pm \mathrm{i}B, \tag{19}$$

where $A = \alpha/2$ and $B = (1/2)\left(h^2\omega_0^2/4 - \epsilon^2\right)^{1/2}$. Hence, solving equations (14) and (15) near the origin is tantamount to determining a dynamical phase-space trajectory in the form $(a, b) = \exp(At)(\cos Bt, \sin Bt)$. We deduce that the equilibrium state is locally stable for $\alpha < 0$, hence $\mathrm{Re}\lambda_{1,2} < 0$, while it is unstable for $\alpha > 0$ (namely $\mathrm{Re}\lambda_{1,2} > 0$). For $\alpha = 0$, the eigenvalues are *imaginary*, so the equilibrium state is a closed orbit centre. We see that the *zero* critical value of the parameter α determines a Hopf bifurcation point, where one encounters a significant qualitative change (a bifurcation) in the system's dynamical profile.

In our case, one physically assumes $\alpha > 0$ (recall that α is defined as a rate of charged dust grain production); therefore, no matter what the initial condition is, the system will never approach the equilibrium state $x = 0$. As a matter of fact, even if it starts very close to the equilibrium state it will keep far from the origin, which therefore *repels* all initial states. The origin $(a, b) = (0, 0)$ determines a *saddle point*, which is unstable. Note that the bifurcation occurs by varying the α parameter.

3.4. Periodic solutions

In order to find the periodic solutions of the system, we have to determine the eigenvalues of the Jacobian matrix of equations (14) and (15), which can be written as

$$\mathbf{J} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where

$$\begin{aligned} a_{11} &= \frac{\partial f}{\partial a} = \frac{\alpha}{2} - \frac{\beta}{8}(3a^2 + b^2), \qquad & a_{12} &= \frac{\partial f}{\partial b} = -\frac{1}{2}\left(\epsilon + \frac{h\omega_0}{2}\right) + \frac{\beta}{4}ab, \\ a_{21} &= \frac{\partial g}{\partial a} = \frac{1}{2}\left(\epsilon - \frac{h\omega_0}{2}\right) - \frac{\beta}{4}ab, \qquad & a_{22} &= \frac{\partial g}{\partial b} = \frac{\alpha}{2} - \frac{\beta}{8}(3b^2 + a^2). \end{aligned} \tag{20}$$

The characteristic polynomial of the above matrix can be expressed, for convenience, as

$$p(\lambda) = \lambda^2 - T\lambda + D, \tag{21}$$

where $T \equiv a_{11} + a_{22}$ and $D \equiv a_{11}a_{22} - a_{21}a_{12}$ are the *trace* and the *determinant* of the Jacobian matrix, respectively. Here, a stable solution can only exist if $T < 0$ and $D > 0$, thus both of the eigenvalues need to be negative. This yields one of the critical boundaries, which is determined by $T = 0$, while $D > 0$. The critical boundary is characterized by a pair of purely imaginary eigenvalues

$$\lambda_{1,2} = \pm \mathrm{i}\sqrt{D}. \tag{22}$$

We see that the periodic solution may lose its stability via a Hopf Bifurcation at the critical boundary, i.e. where $T = 0$.

4. Numerical results and discussion

The Van der Pol–Mathieu equation possesses an oscillatory (periodic) solution, which is a *periodic attractor*: every nontrivial solution tends to this periodic solution. Periodic solutions may be sought by varying α and β parameters.

For the purpose of integrating the Van Der Pol–Mathieu equation (12) numerically, the latter may be expressed as a set of two coupled ODEs in the form

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= y, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= (\alpha - \beta x^2)\frac{\mathrm{d}x}{\mathrm{d}t} - \omega^2(t)x. \end{aligned} \tag{23}$$

We may now investigate the dynamical profile of equations (23) numerically. We shall use an indicative set of fixed values for the system parameters: $\omega_0 = 1.0$ and $h = 0.01$, in addition to the initial conditions $x_0 = y_0 = 1.0$ for $t_0 = 0$. Employing a fourth-order Runge–Kutta method, we have solved equation (11). The system was found to possess various stable and unstable limit cycles. The phase diagram in (x, y) and (t, x) planes, for different values of α and β, is depicted in figures 2–4. Periodic states occur when we choose $a_{11} + a_{22} = 0$, i.e. $\alpha = \beta$; see in figure 1. For $\alpha < \beta$, the system exhibits a stable limit cycle: large amplitude

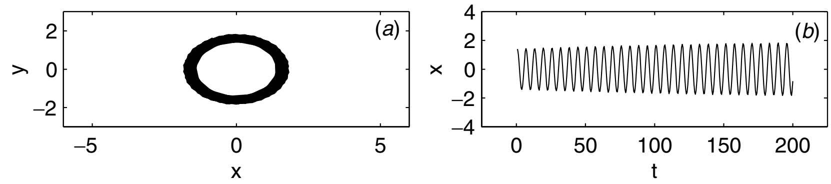

Figure 1. Phase diagram of (a) x–y and (b) t–x planes for $\alpha = \beta = 0.01$.

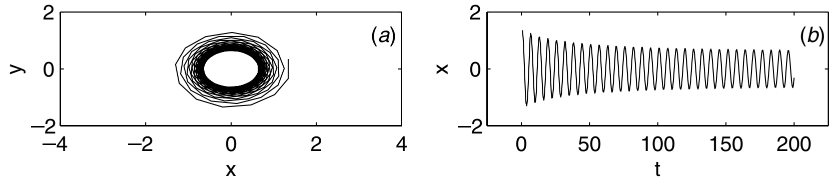

Figure 2. Phase diagram of (a) x–y and (b) t–x planes for $\alpha = 0.01$ and $\beta = 0.1$.

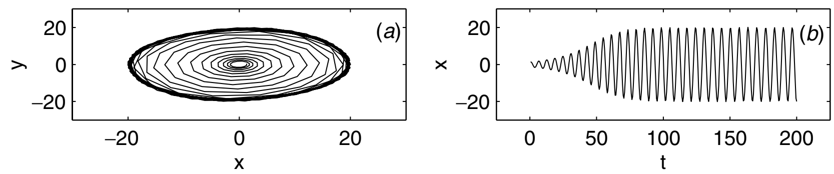

Figure 3. Phase diagram of (a) x–y and (b) t–x planes for $\alpha = 0.1$ and $\beta = 0.001$.

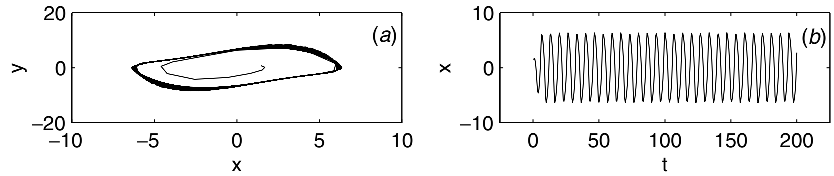

Figure 4. Phase diagram of (a) x–y and (b) t–x planes for $\alpha = 1$ and $\beta = 0.01$.

initial states are attracted to the limit cycle (cf figure 2). In figure 3 (for $\alpha = 100\beta = 0.1$), the system's behaviour is initially unstable, and a typical chaotic limit cycle picture is obtained; the solution later tends to a limit cycle from inside. As α increases, the system attains a stable state in a deformed limit cycle (cf figure 4). The stability profile may be investigated in terms of the amplitudes (a ad b) and parameters ϵ and h. The stability region in the (α, β) plane is represented in figure 5 (for a representative set of amplitude and parameter values.

The Van der Pol oscillator (with no external force) is known to converge to a limit cycle and tracks a stable orbit. In our case, we see that lower values of α lead to a limit cycle similar to that of the (stable) Mathieu equation, while for higher values of α, a profile similar to the limit cycle of the Van der Pol equation is recovered. We conclude that this hybrid evolution system features a balance among an instability region, where it behaves according to

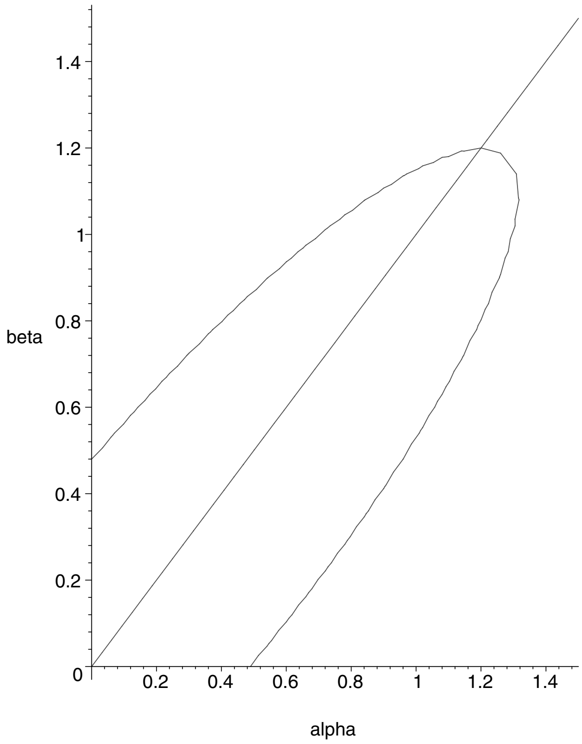

Figure 5. Stability and instability regions in the ($\alpha - \beta$) plane, for $\epsilon = 0.1, h = \omega_0 = 1$ and $a = b = 1$ (cf equations (20) and (21)). The regions enclosed *below* the line $T = 0$ (i.e. for $T < 0$) and the oval-shaped curve $D = 0$ are stable; all other regions are unstable.

the Mathieu equation, and a stability region, where it follows the Van der Pol equation profile (recall that the VdP equation always possesses a periodic solution).

5. Conclusion

We have derived a nonlinear Van der Pol–Mathieu-type evolution equation (11) for the density of dust grains in dusty plasmas by considering an appropriate source term in the dust density equation. The solution of this combined Van der Pol–Mathieu-type equation depends on the relevant physical model parameters, namely the production rate α and the loss rate β. We note that the system described by equation (11) is linearly unstable near the origin, for small-amplitude perturbation, so orbits grow as they are repelled by the origin $x = 0$. The nonlinear dissipative term $\beta x^2 \mathrm{d}x/\mathrm{d}t$ in equation (11) eventually limits this growth, and the amplitude saturates. However, due to the third term in equation (11), the evolution of any known modulated oscillator function near the natural frequency of our system is controlled by (and competes with) the loss term β. Varying the values of α and β, the behaviour of the solution to this equation exhibits a variety of interesting profiles.

The physical interpretation of these novel results is straightforward. We see that the charge residing on the surface of dust grains, although initially zero (neutral dust), grows as a result of random perturbation and abandons the vicinity of the origin quite fast. A dynamical steady state in the form of a limit cycle is then reached, whose characteristics depend on the physical parameters involved. This state behaves as an attractor with respect to nearby states. The well-known phenomenon of dust grain charge reaching an asymptotic value via random charge oscillations, as observed in experiments [1], is thus formulated in a different framework, via the sketch of a nonlinear charge dynamical profile. Naturally, setting rates α and β to zero, the ordinary dust-acoustic oscillations are recovered (assuming a constant initial dust charge).

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